Technology
and the
Yin&Yang
of
Teaching and Learning Mathematics

The essence of using technology,
in particular computer algebra systems (CAS), in education

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Abstract: *We develop a model comprising six teaching and learning archetypes and use this model to look at the various roles that technology, in particular computer algebra systems (CAS), can play for each.*

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Preface

This is a summary of my academic work of the past twenty years. I dedicate this text to two giants who let me stand on their shoulders: Bruno Buchberger and David Stoutemyer. Their influence on my work was enormous. Thank you!

Bernhard Kutzler, November 2008

Introduction

Humans are ruled by two forces: *Hold on* and *Let go*. These two forces correspond to *Yin* and *Yang*, the two elementary polar energies that are considered and studied in Eastern philosophies. In the context of teaching and learning mathematics and related subjects these two energies manifest as *Connect* and *Automate*. *Connecting* is an active, seeking form of *holding on*. *Automating* means to let a tool do what we used to do ourselves (such as performing arithmetic operations), i.e. we *let go* these tasks.

Hold on	**-**	***Yin***	**-**	**Connect**
Let go	**-**	***Yang***	**-**	**Automate**

A car is a tool for automating transportation. Instead of walking to the grocery shop, we can go there by car. This saves us from having to walk between our home and the shop and from having to carry the groceries. For some people, using a car for their shopping is a *convenience* that saves time and energy that they can then use for other activities – such as reading a book. For people who are physically challenged, using a car for their daily shopping may be a matter of *survival*.

This example shows two motivations for automation: *Amplification* and *compensation*. Here is another example: Optical instruments such as telescopes and microscopes *amplify* our natural eye sight so that we can see things that we cannot see otherwise. Optical instruments such as eye-glasses *compensate* poor eyesight so that people with poor eyesight can see things that people with normal eyesight can see without glasses.

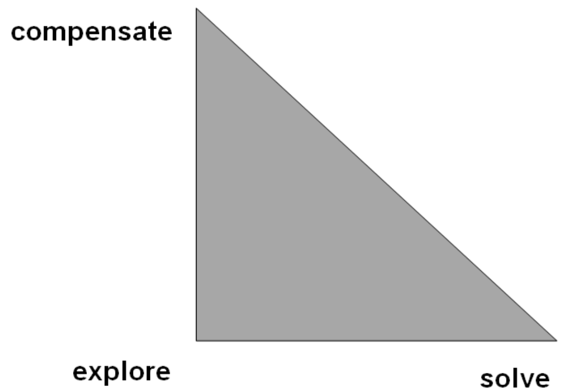
This can be further refined. *Amplification* in itself has two aspects based on the motivation to amplify. One can use a telescope to look at a distant object as may a private detective or a policeman do when observing a suspect – or as may an astronomer do when observing a moon eclipse. Alternatively one can use a telescope to scan the sky in the search for new stars. These two uses may be named *solving* and *exploring*. Likewise with the car example: Using a car for shopping or for visiting a friend who lives in another city *solves* a transportation problem.

Driving a car around California on a holiday trip is a nice way to *explore* the most populous U.S. state.

This gives three automation archetypes based on the motivation to automate:

Automate = Compensate + Amplify
= Compensate + Solve + Explore

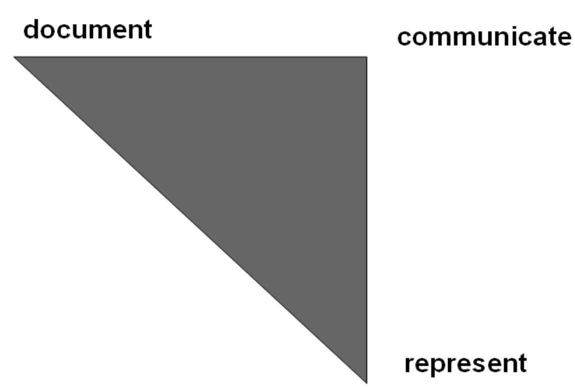
Here is a visualization of the *Automate* triangle[1]:



Connection also comprises three archetypes based on what to connect with what, notably *representation*, *documentation*, and *communication*. *Representation* is about connecting models with models, such as connecting an algebraic model (an expression) with a graphic model (a graph) or a numeric model (a table). *Documentation* is about connecting models with humans, such as writing a paper on how a problem was solved. *Communication* is about connecting humans with humans, such as having students work in pairs or groups.

Connect = Represent + Document + Communicate

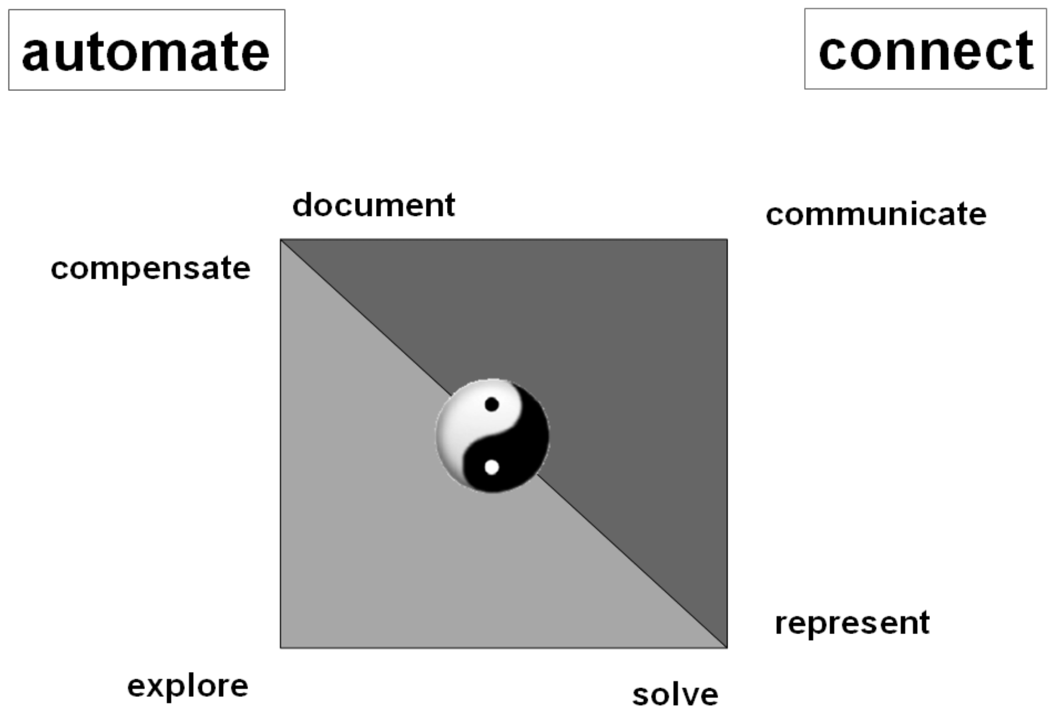
Here is a visualization of the *Connect* triangle[2]:



[1] The *Yang* triangle usually points upwards.

[2] The *Yin* triangle usually points downwards.

Putting these two triangles next to each other yields a picture that I call the *Yin & Yang of Teaching and Learning Mathematics*:



This picture shows six archetypes that we encounter in the context of teaching and learning mathematics (and related subjects). The benefit of this model is to allow for a better understanding of how to best integrate technology into mathematics education.

Before we go through each of the six archetypes and discuss the various roles that technology, in particular computer algebra systems (CAS), can play for each we present an additional picture that is helpful for understanding the benefit of using technology for both mathematics and mathematics education.

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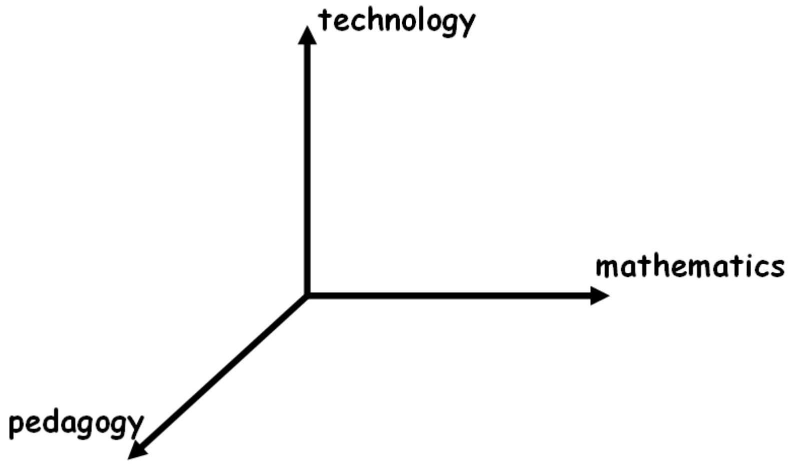
Mathematics/Pedagogy/Technology-Space

Let the *x*-axis of a three dimensional system of orthogonal coordinate axes represent mathematics, the *y*-axis represent pedagogy, and the *z*-axis represent technology.



In this model, mathematicians are people who work “on the mathematics axis”, i.e. in linear 1D “space”.

Mathematics teachers are people who work “in the mathematics/pedagogy plane”, i.e. in planar 2D “space”. A mathematics teacher has to know mathematics *and* the pedagogy of how to teach a person some mathematics.

Say, a person has a mental capacity of 100 “units”. As a mathematician, this person can use this capacity for a “professional span” of a length of 100 units on the mathematics axis. As a mathematics teacher, this person can use the same capacity for a “professional span” of an area of 100 units in the mathematics/pedagogy plane, which corresponds to, for example, a rectangle measuring 20 units along the mathematics axis and 5 units along the pedagogy axis.

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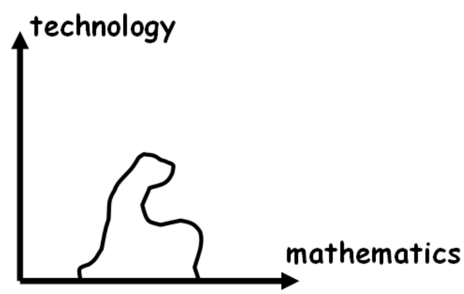
Traditionally mathematicians used simple technology such as paper and pencil to amplify their brain power for performing mathematical tasks. Calculation tools such as abaci (abacuses) are in use for many thousand years already, the Sumerian abacus dates back to 2700-2300 BC. Abaci facilitate the performing of arithmetic operations. The basic operation on an abacus is to add or subtract one. Basically, today’s computers are very much advanced abaci with sophisticated electronic mechanisms built on top of the basic operations[3] in order to perform increasingly complex tasks that today include graphing, dynamic geometry, computer algebra, and theorem proving.

Computers have greatly changed the world in general – and the world of mathematics in particular. One of the first significant uses of a computer for mathematical research was the proof of the Four Colour Theorem[4]. Some mathematicians still don't accept the proof arguing that a human cannot verify it in practice[5]. However, computing the number Pi to millions of digits or finding very large prime numbers also requires a computer and a human verification is far beyond being practical or realistic. Should we not accept these results and should we not use very large prime numbers in bank or internet security systems just because a human cannot verify it?[6]

More and more mathematicians accept computer software, in particular powerful numeric, graphic, and algebraic software environments, as tools for mathematical research. These mathematicians move from the 1D mathematics axis into the 2D mathematics/technology plane. A "visual argument" for the benefit of this is that one has infinitely many more ways of connecting two points on the mathematics axis by allowing paths in the mathematics/technology plane. The higher that we can go on the technology axis, the more paths are possible, enabling solutions and findings that are not possible without (or with less powerful) technology.



[3] The basic language of a computer is the language of binary numbers, i.e. sequences of zeros and ones, which somehow resemble patterns of pebbles of an abacus. Basic operations on binary numbers compare to shifting pebbles of an abacus.

[4] The Four Colour Theorem states that any plane separated into regions can be coloured using no more than four colours such that no two adjacent regions have the same colour. (Two regions are adjacent iff they have a segment as a border.) Political maps are typical examples.

[5] It goes without saying that a human could verify the proof in theory by just executing the computer program step by step, using a lot of paper and a lot of pencils. But probably it would take several human life times to perform the proof manually.

[6] Such a traditionalist view can be a stumbling block for further progress in any area. When automobiles started to be used, some people were afraid of using them, saying that moving at a speed faster than walking would be dangerous. Where would we be without technology?

Technology makes some mathematics possible.
(Bert Waits)

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The same argument holds for accepting computer tools for teaching and learning mathematics. Mathematics teachers using technology move from the 2D mathematics/pedagogy plane into the 3D mathematics/pedagogy/technology space, which means infinitely many more possibilities of connecting two points in the mathematics/pedagogy plane by permitting 3D paths. This is like allowing helicopter trips in a landscape that previously could be explored only using “surface-attached” tools such as bikes, cars, or ships.

Gifted students may be able to have their brilliant minds fly from A to B – but what about the other students? A ride on the “mathematical helicopter” may be what they need for the trip!

I had a very touching experience once with a group of students. As part of a teacher training course I taught a group of students while the teachers observed the lesson. We used the (legendary) TI-92 handheld and I let the students do some work in analytic geometry. I asked the students a quite demanding question and told them what kind of experiments they should do on their handhelds in order to find the answer. I walked through the classroom to see how well the students did and after a while I saw the first students succeed. Suddenly a girl shouted: “Yes!” I encouraged her to share her findings – and she gave a perfect answer. After the end of the lesson her teacher told me that she was his “weakest” mathematics student. But in my class she was as fast and as successful as the best of her classmates. For her the use of technology made a big difference!

Technology makes some mathematics pedagogy possible.
(Bert Waits, extended)

*Not the **tool**, but the **use of the tool** is or is not pedagogical.*
(Vlasta Kokol-Voljc)

In the following chapters we will look at the pedagogical motivations for doing mathematical helicopter rides.

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The three axes model also lets us better understand how to best do technology training for teachers. Novice teachers have to learn how to fly the mathematical helicopter, they have to learn how to use the helicopter to transport loads and passengers, and they have to learn how to instruct students to do certain helicopter manoeuvres. (In fact, most students learn very quickly how to fly the mathematical helicopter.)

It is a frequently observed mistake to do technology teacher training with “interesting” or even “challenging” mathematics – or to try to “sell” new pedagogical ideas in an introductory technology training course. This is like introducing students to the technique of differentiation by doing a challenging optimization problem – just to also show them how useful differentiation is. Gifted students may be able to digest such a “heavy mathematical meal”, but the majority of students will suffer from “mathematical indigestion”. Likewise with teachers: technology lovers may be able to handle a steep learning curve with learning technology and new mathematical and pedagogical opportunities at the same time, but the majority of teachers will need a gentle ride first along the technology axis, then into the mathematics/technology plane, and only finally into mathematics/pedagogy/technology space. Teacher training should be done gradually and with as much (pedagogical) care as any kind of teaching.

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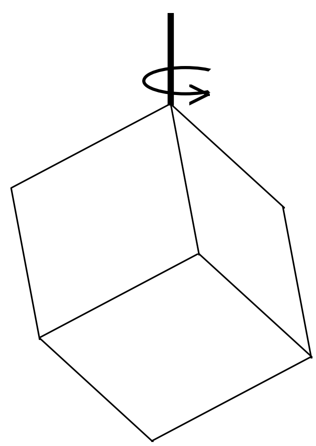
Represent

Consider the following problem: A homogeneous[7] cube hangs by a thread attached to one corner. It is otherwise free to move. When we look at this configuration from the front (as in the following picture), then we see the cube as three parallelograms.

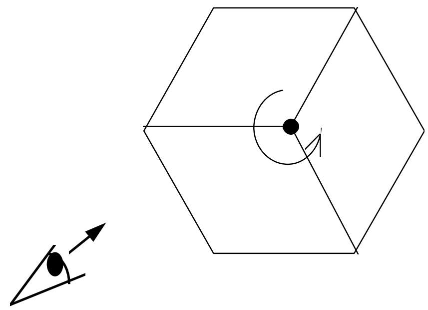
What is the smallest angle (greater than zero) through which we must spin the cube so that we see the same figure?



Trivially we get the same figure after rotating the cube a full 360 degree, but does it happen earlier?

[7] If the cube is not homogeneous, its centroid may not lie in the intersection of the three spatial diagonals. However, this is what we want to assume here.

We vary the representation of this problem in that we use a new point of view. We look from above, where the thread attaches. Then we see the following figure. (The previous eye position is also shown.)



From this point of view the answer to the posed question is obvious. For a rotation of 120 degree, 240 degree, and 360 degree the figure remains the same, so 120 degree is the solution to the problem.

The problem appeared demanding when looking at the first picture. The solution is obvious when looking at the above picture.

If you have a problem, there are two paths open to you:
either you solve the problem, or you change your view.
(Chinese Proverb)

Through changing the view in the above example, the solution became obvious. Changing the view means changing the representation.

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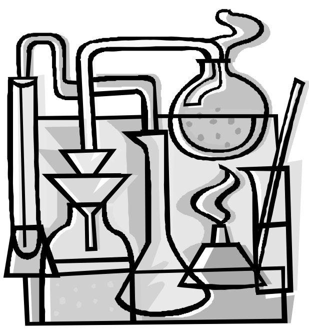
Representations play a central role in mathematics. Various representations are like various points of view. A city appears completely different when viewed from above, perhaps from the basket of a hot-air balloon, to how it does when viewed from a neighbouring hill, and different again when viewed by someone taking a walk around the city itself.

If a question about an object is posed, one should take a point of view that makes it as simple as possible to find the answer. To find the quickest way from the council house to the city gasworks one uses the hot-air balloon point of view (that we obtain in the form of a city map). The question as to the tallest building in the city is easily answered from the neighbouring hill, while the colour of the main door of the church is most easily answered from within the city itself.

One of the basic techniques of mathematical problem solving is to find a representation of a problem that makes the problem easy to solve, if not actually making the solution obvious. Therefore one can look at mathematical problem solving as the *art of transforming representations until the solution is visible*. The problem with the hanging cube is a fine example. Another example is the solving of an equation such as $5x-6 = 2x+15$ by transforming it into the equivalent equation $x = 7$. Both equations define the number 7, hence both equations can be considered a representation of 7. The first equation, $5x-6 = 2x+15$, is an implicit representation of 7. The second equation, $x = 7$, is an explicit representation of 7.



In chemistry we use the method of distillation to obtain the essence of, for example, a plant. If we carefully distil peppermint leaves, we obtain the essential oil of peppermint. Solving an equation is a similar process. The method of solving an equation by applying equivalence transformations can be seen as a distillation process for obtaining the equation's *essence*, which in mathematics is called the equation's *solution*.

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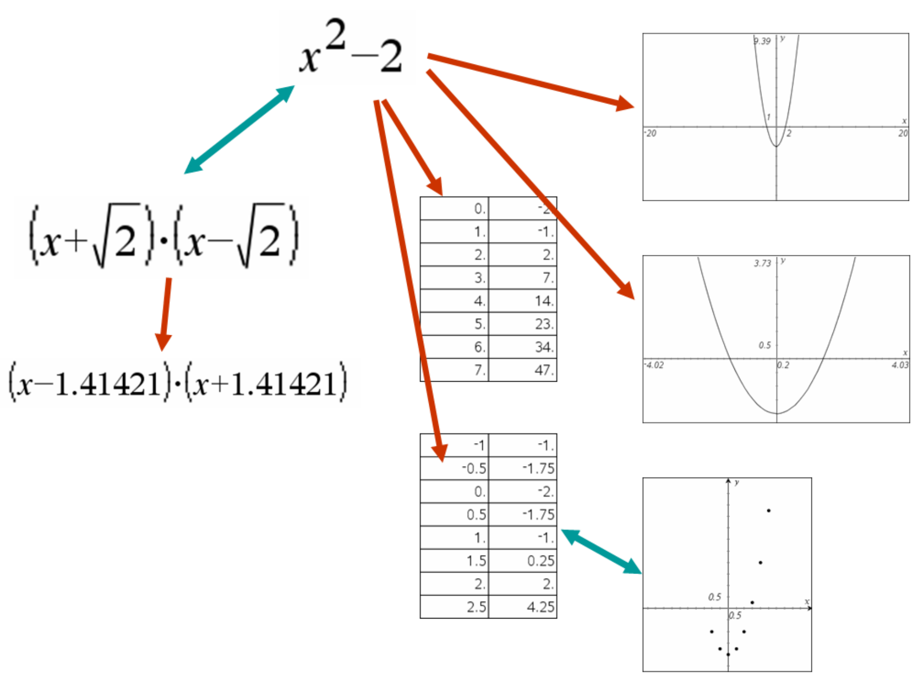
There are many kinds of changing a representation. One can change the representation type, such as going from algebraic to numeric or graphic. One can change the "point of view" within a representation type, such as going from an expanded form to a factored form, from a table with starting value 0 and increment 0.1 to a table with starting value 1 and increment 0.5, or from a graph with a certain plot range to a graph with a different plot range.

What we consider a *calculation* (or a *simplification*), such as rewriting '1+2' as '3' or rewriting '3a+4a' as '7a', also can be seen as a change of representation. Both '1+2' and '3' represent the same thing and so do '3a+4a' and '7a'.

Rewriting $\sqrt{24}$ as $2\cdot\sqrt{6}$ is a change of representation. Approximating $\sqrt{24}$ to the decimal fraction 4.89898 is a change of representation. But there is a difference with the latter: while one can go "back" from $2\cdot\sqrt{6}$ to $\sqrt{24}$, because these two representations are equivalent, one cannot go back from 4.89898 to $\sqrt{24}$, because information got "lost" when going from $\sqrt{24}$ (which, written as a decimal fraction, has an infinite number of digits after the decimal point) to 4.89898. Nevertheless there are situations when an approximation is more useful than the precise original expression, such as when answering questions involving order.

Loosing information when approximating $\sqrt{24}$ by a decimal number has an interesting aspect: On a scientific calculator, 4.89898 is the *only* way of "representing" the square root of 24. Most calculators can represent simple rational numbers such as one third only as decimal fraction approximations with a certain number of digits (such as 0.33333333). Scientific calculators are materializations of number sets R(n), where R(n) denotes the set of decimal fractions with n digits. R(n) is a true subset of the set of rational numbers $\mathbb{Q}$. These number sets R(n) are *not* closed with respect to multiplication or division, hence simple identities such as $\frac{1}{x} \cdot x = 1$ or $x^2 - y^2 = (x+y) \cdot (x-y)$ may not be valid.[8]

The following picture shows various representations of the algebraic expression $x^2 - 2$.



0.	-2.
1.	-1.
2.	2.
3.	7.
4.	14.
5.	23.
6.	34.
7.	47.

-1	-1.
-0.5	-1.75
0.	-2.
0.5	-1.75
1.	-1.
1.5	0.25
2.	2.
2.5	4.25

Double headed arrows indicate equivalence, i.e. full preservation of information, so that one can move in both directions. Single headed arrows indicate loss of information, so that one can move only in one direction.

[8] $(1/x) \cdot x = 1$ is not valid in any R(n), although most calculators hide this in obvious cases, for example by rounding 0.999999999 to 1.

For $x^2 - y^2 = (x+1) \cdot (x-1)$ look at R(1) and choose x=1.1 and y=0.2.
The left hand side gives $1.1^2 - 0.2^2 = 1.2(1) - 0.0(4) = 1.2$.
The right hand side gives $(1.1+0.2) \cdot (1.1-0.2) = 1.3 \cdot 0.9 = 1.1(7) = 1.1$.

$(x+\sqrt{2})\cdot(x-\sqrt{2})$ is the factored equivalent of x^2-2 and one can "go back" by expanding the expression. $(x+1.41421)\cdot(x-1.41421)$ is a decimal approximation of $(x+\sqrt{2})\cdot(x-\sqrt{2})$, so these two expressions are not equivalent. The upper table in the above picture is a function table for x^2-2 with starting value 0, increment 1, and end value 7. The lower table is a function table for x^2-2 with starting value -1, increment 0.5, and end value 2.5. Such tables are a significant reduction of information. One cannot go back from such a table to the original algebraic expression. Also, the lower table cannot be obtained from the upper table; in order to produce it one has to start over with the algebraic expression x^2-2. Graphs are geometric equivalents of function tables obtained via the Cartesian coordinate concept. The lowest graph is an equivalent of the lower function table. It is a discrete *scatter plot* showing the eight points whose coordinates are in the table. The upper two graphs are continuous *function plots*. In fact, they *appear* continuous, very much like a movie appears continuous because our eye cannot make out the many discrete pictures it comprises of. A function plot is obtained by evaluating the function at very many values of *x*, often only a pixel size apart from each other, and connecting neighbouring graph points. Therefore, their function table equivalent would be very long tables with very small increments.

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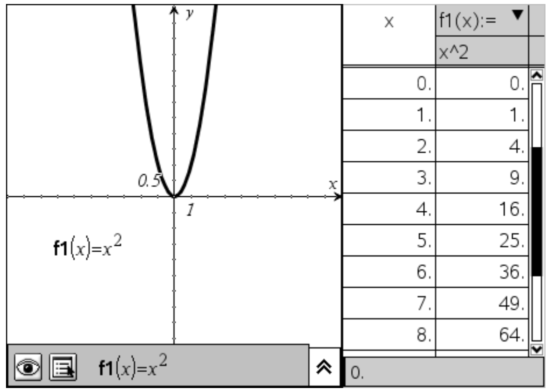
One of the core skills of a mathematician is to simultaneously hold different representations of a (mathematical) object in his or her mind and to choose the one that is most useful in a given context.

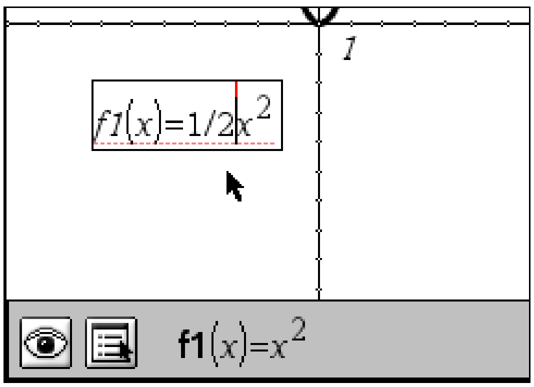
Mathematics teachers strive to help their students develop this skill. While one cannot have the students "look into a mathematician's brain", one can employ technology to simulate a mathematician's mind. Seeing several representations of an object on the screen right next to each other and seeing how all other representations change when one representation is modified is an extremely powerful pedagogical approach that is possible only with computer technology.

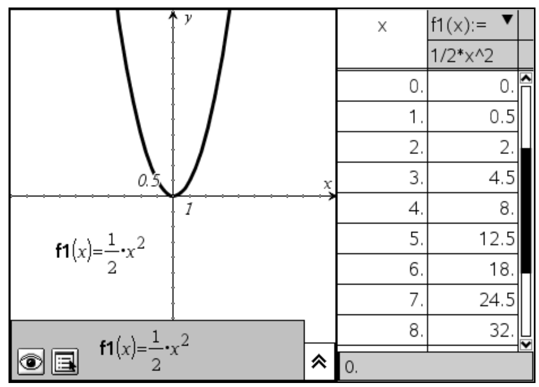
Therefore a good mathematics teaching and learning tool should offer an easy way of changing representations, i.e. to switch between models (algebraic, graphic, numeric) and to have all these representations be linked dynamically.

The following picture shows a TI-Nspire[9] screen with the function expression $f1(x) = x^2$, a corresponding function graph, and a corresponding function table.

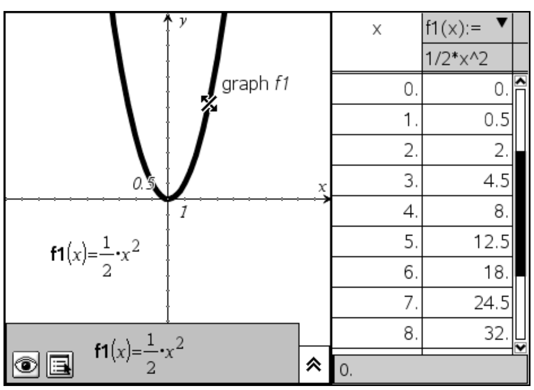


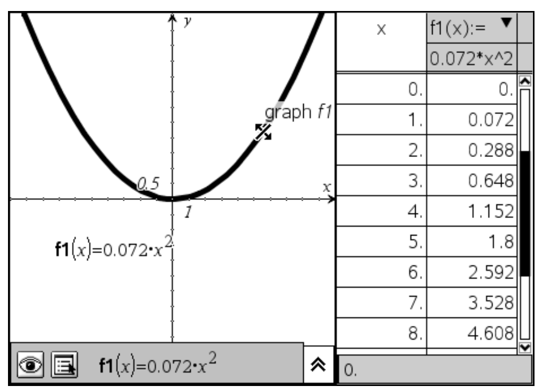
One can edit the function expression, and then observe how both the function graph and the function table are updated automatically.





One can grab the function graph, drag it, and then observe how both the function expression and the function table are updated automatically.





[9] TI-Nspire is a powerful mathematics tool produced by Texas Instruments. It comprises graphing, interactive geometry, a spreadsheet, interactive statistics, a text editor, a program editor, and a data collection application. TI-Nspire CAS also includes computer algebra.

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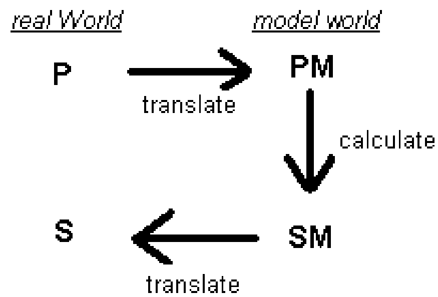
Typically, mathematical problem solving is about solving real world problems[10] with mathematical methods. Characteristic for mathematical problem solving are the three steps shown below.



The first step is choosing the model and translating the real world problem P into the language of the model, yielding the model problem PM. This is the entrance into the world of mathematical representations. And this is where the mathematical "alchemy" starts with its art of transforming representations until the problem's essence – its solution – is found.

Going from a real world representation[11] to a mathematical representation requires to grasp and understand[12] the situation, it requires to know a large enough "toolbox" of mathematical representation types, also called "mathematical models", and it requires to be able to choose an appropriate representation/model from this toolbox.

An optimization problem, for example, may translate into a function to be optimized and equations that describe constraints between the variables.

The second step is applying the available algorithms to solve the model problem PM (= to transform the mathematical representations), yielding a model solution SM.

The third step, finally, is to translate the model solution SM into a real world solution S.

However, now we still need to test, if S actually is a solution of P. If it is not, then the whole process needs to be repeated, because the mistake or error could be any-

[10] Most problems used in the classroom are "idealized" real world problems that may better be named "quasi real world problems" or "near real world problems".

[11] This might be a real world situation that one observes or a natural language description of a real world situation.

[12] See the chapter *Document* for more on understanding.

where: The chosen model may be inappropriate, the translation may be faulty, or there may be an error in the calculation.

Traditionally, problem solving is treated at school only half-heartedly. Main emphasis is on the second step, calculation, and its execution with paper and pencil. Typically one can do, may be, three optimization problems in an one hour lesson using up to 80 % of the time for (hand) calculations. Only about 20 % of the time may remain for mathematical modeling. Hence, most problem solving exercises turn into exercises for practicing the required calculation skills. And this we call “problem solving training”?!

Choosing models and translating from the real world into mathematics and vice versa rarely are taught explicitly. Therefore it is understandable that a majority of students don’t develop this ability. Hence, they are afraid of exercises requiring such translations. With the (extensive) use of powerful technology such as CAS for the calculation step[13], one can dedicate a lot more time to teaching the choice of models and how to translate real world problems into the language of mathematics. One may be able to treat ten or more optimization problems in an one hour lesson spending 80 % of the time on modeling and only 20 % on calculations. This would be the proper “problem solving training”, an important part of which is learning to find a mathematical representation of whatever has to be solved.

By employing technology as widely as possible, we can dedicate enough time to teach the choosing of mathematical models and the translating into the language of these models. Once these skills are taught explicitly, more students will appreciate and master them.

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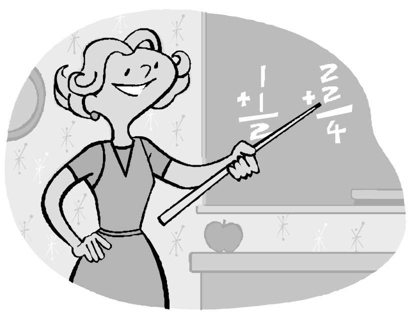
[13] See the chapter *Solve* for more on calculating.

Document

The word *document* derives from the Latin word *docere* = *teach*. Therefore, a document is a paper (more generally: an object) that teaches something or, in other words, that proves something.

When you solved a (mathematical) problem and you want to prove – to whomever – that you did, you have to document your solution and, eventually, the method that you used to find the solution. If you are a student, you may have to prove to your teacher. If you are an employee of a company, you may have to prove to your boss. If you are a freelancer, you may have to prove to your customer. If you are a scientist, you may have to prove to the academic world. Even if none of the above applies, you may want to prove to yourself at a later point in time, i.e. after you may have forgotten the thought process that you just went through.



By its very nature, documenting requires the ability to *argue*, i.e. to *convince* somebody with a certain level of knowledge (that level of knowledge should be lower than the knowledge that you have right after solving the problem). The ability to argue, in turn, requires the ability to *design* the content of the document (for which *creativity* is needed) and the ability to *describe* the content using natural language and pictures (with graphs, tables, sketches, …), i.e. to convey a message with an intended meaning in an unambiguous manner.

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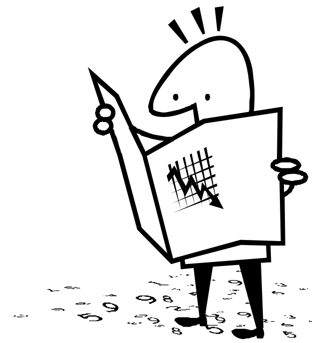
Describing is the inverse of *understanding*, which is the skill to interpret a given natural language text or document. Describing and understanding are very closely related and depend on each other. Therefore these skills should be developed and trained together. Because a text can have multiple meanings or even be contradictory, comprehension requires the ability to recognise plurality of meaning or contradiction and, where necessary, to look into each possible interpretation. A good example is the sentence *"I saw the man on the hill with the telescope"*, which has several possible interpretations with variations of who is on the hill and who has the telescope.

As simple as it usually is to understand a short sentence, a longer sentence or text can be almost incomprehensible, for example the instructions for a video recorder, an insurance policy, a law statute – or a word problem in a mathematics book.

While we are on the subject: as already discussed at the end of the previous chapter, word problems can and should be used for practising and training the understanding skill, the modeling skill, and the translating skill (they are all part of the problem solving skill). Word problems should not be misused for practising the calculation skills that are required to compute the solution.

A valuable and quite useful practice is the understanding of expert opinions. What does it mean when an expert says, *“There is no proof of environmental damage caused by this installation”*? It does not mean that it is clean (even if the manager of the facility wants to interpret it this way). Neither does it mean that the installation is dangerous (even if the protesters want to interpret it that way). For this type of exercise every daily newspaper is full of examples waiting to be used for practicing both *understanding* and *describing*. After a text has been understood, a new formulation can be sought that is shorter, clearer, less ambiguous, …



To begin with, newspaper articles can make a fine source for improving students’ skills of understanding and describing. For a more formal approach one can deal with simple mathematical logic, where students translate between everyday language and the language of mathematics. Following are a few examples. Going from left to right practices *understanding*. Going from right to left practices *describing*.

27 is divisible by 3	⇦⇨	$\exists t(27 = t \cdot 3)$
Every number is greater than its predecessor	⇦⇨	$\forall x(x > x-1)$
The square of a number is non-negative	⇦⇨	$\forall x(x^2 \geq 0)$
There is a number with its square equal to 4	⇦⇨	$\exists y(y^2 = 4)$

☯

Argumentation is about finding convincing reasons for something. In order that they are convincing, they must, among other things, be consistent with the state of knowledge of the listener/reader. For instance, the arguments about the necessity of a certain medical therapy are different if the doctor is talking to a medical colleague rather than to the distressed, uneducated father of a child.

Argumentation is the first step towards the very important mathematical skill of constructing proofs. Exercises of the following kind serve to train this skill:

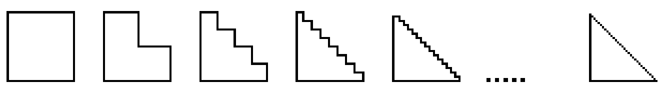
- Explain why one uses the first derivative to find the extreme values of a function.

- If a curve is symmetric in both the *x*- and *y*-axes, is it then symmetric through the origin? Give the reason or find a counter-example.

The important understanding that even a thousand examples are not proof of a statement, while one counterexample is enough to disprove it, could be developed through various case studies.

Furthermore the danger of accepting an argument too quickly should be demonstrated. The following example is very illustrative and, for educational purposes quite useful.

> A square has side length 1. Thus the sum of the lengths of the upper and right hand sides is 2. Making a single step, as shown below in the second picture from the left, does not change the total length; the resulting step thus has also length 2. Cutting further steps as shown in the third and further pictures below does not change the total length, so we see that all such steps have length 2. If we proceed, then the limiting case is the diagonal, which must, therefore, have length 2. Since we already know that the diagonal has length $\sqrt{2}$, we deduce that $2=\sqrt{2}$.



Recreational mathematics books are full of such examples. We need only the courage to introduce them to normal teaching situations. School mathematics would, especially in the eyes of many of the students, lose much of its dryness.

☯

Documenting is an important part of mathematics education. It gains importance in traditional mathematics classroom activities, namely within the productive phase of applying mathematical knowledge to solve problems.

In traditional assessments, the solution to the problem is often the only goal, and the craftsmanship of performing the calculations that are required to obtain a solution earned a student a good mark. This changes with technology, in particular powerful technology such as CAS, because finding a solution of an equation or finding an integral is a different kind of work when all you need is to press the appropriate keys or do the appropriate mouse clicks. It still is *mathematical* work because one can undoubtedly argue that choosing the appropriate sequence of commands or keys from a more or less large selection of keys and commands does require mathematical know how. However, this kind of mathematical work requires much less time than performing the underlying calculations with paper and pencil.

The result loses importance because it is easy to obtain. A documentation of how the problem was solved is much more than just a good replacement: documentation of experimenting or problem solving provides valuable feedback in the teaching process, both during the concept development phase and for the assessment.

For assessment, documentation of mathematical experiments or mathematical problem solving is comparable to a composition written for an English class. The documentation of mathematical work is not just right or wrong. It can be too short, too long, incomplete, or it can deviate from the subject, etc.

☯

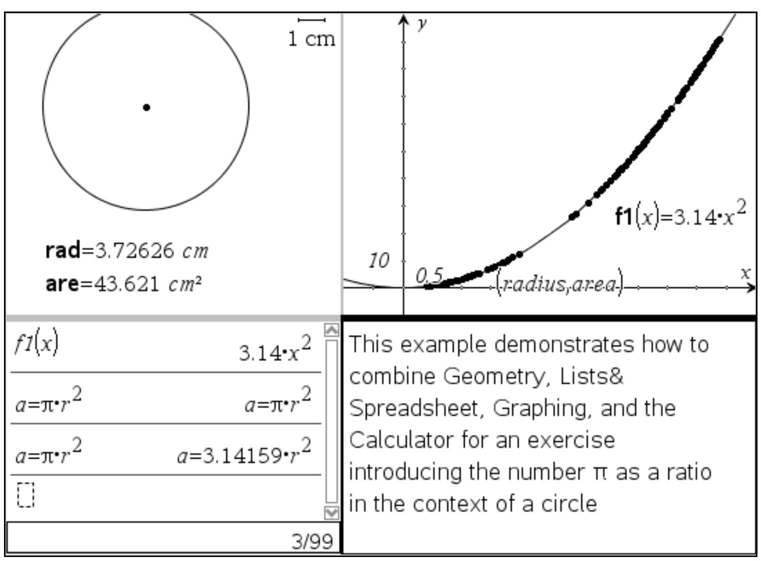
Technology has two roles for the *document* archetype:

(1) The presence of technology means a shift of focus from executing algorithms to documenting mathematical work, i.e. it strengthens the role of documentation in the classroom.

(2) Technology can support the production of documents. A good mathematics teaching and learning tool should offer an easy way to create documents, using text and appropriate images of the changing representations used to solve a problem.

☯

The following screen image shows a document that was produced with TI-Nspire CAS.



Communicate

Communication is the most natural mental activity of a human. The word *communicate* derives from Latin *com* = *together* and **moinicos* = *carrying an obligation* (from *munia* = *duty*, *obligation*) and means the imparting or transmitting of ideas, knowledge, information, etc. It is related to the word *community*, hence reflects a central aspect of the human as a social being.

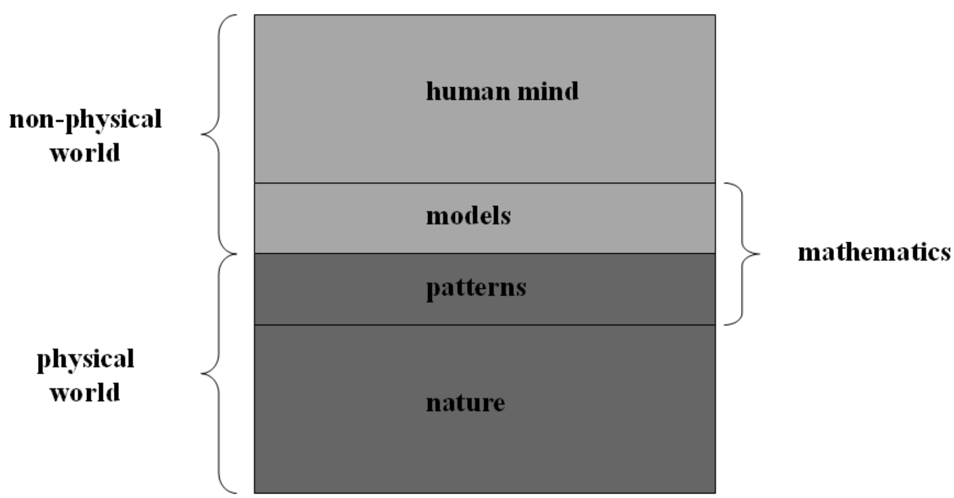
The means of communication is *language*. Humans communicate with each other using natural languages such as English as well as body languages. In our context we only look at the spoken or written language.

Mathematics also is a language.

The book of nature is written in the language of mathematics.
(Galileo Galilei)

While natural languages connect humans with humans, mathematics connects humans with nature.

Many contemporary mathematicians consider mathematics *the science of patterns*. We look at phenomena around us and observe patterns. An example is the fact that an object that is released moves towards the centre of the earth until the movement is stopped by an obstacle such as a table or a floor. This pattern was named the *law of gravity*. Nature communicates with us via such patterns. Mathematics can be seen as the "interface" between the physical world of nature and the non-physical world of the human mind.



The word *mathematics* derives from the Greek word *mathema* = *learning, knowledge* from *manthanein* = *to learn*. Therefore, mathematics originally meant all systematic collections of knowledge, i.e. all kinds of *sciences* (from Latin *scientia* = *knowledge*).

Using the notions of information technology, there are two kinds of knowledge connected with nature: (1) the programming language in which nature is “written” and (2) the program code of nature.

By trying to “reverse engineer” nature we aim at learning about both the program code and the programming language. While originally all this was seen as *mathematics*, nowadays we use *mathematics* only for the abstract form of knowledge (the “programming language” of nature) and the study of the “material” form of knowledge (the “hard-wired program code” of nature) is studied as the *natural sciences*. Natural sciences were further split into disciplines such as physics, chemistry, biology, etc.

In the context of the six archetypes we use communication in the narrow sense as a connection between humans.

In fact, documentation can be considered an “offline” form of communication with a possibly anonymous and typically remote communication partner. Therefore, the skills discussed in the previous chapter, i.e. *understanding*, *describing*, and *arguing* are relevant also here.

In addition there are some important issues regarding the “online” aspects of the communication between student and teacher as well as the (learning related) communication among students.

First we look at the communication between student and teacher. Traditionally, teachers almost exclusively use the method of *front teaching*, where the teacher is active and the students passive. The opposite of this would be a classroom setting in which the students work either freely exploring or under the guidance of the teacher, while the teacher acts as an advisor and assistant in case of trouble, as is typical when students use technology. Such teaching situations where the students are active and the teacher (principally) passive, especially encourage the independent activity and the creativity of the students. The ideal teaching methodology lies in a good mixture of these two forms.

Next we look at the communication between students. Traditionally students almost exclusively work *alone*, as is most typically enforced in an exam. However, later in life they may need to work in teams, so *teamwork* should be encouraged and practiced in school already. The use of technology is a good opportunity to have students make explorations or solve problems in teams.

A team is a group comprising at least two people. An argument that is often used against teamwork is that the work in a team often is done by only one or a few gifted students, while less gifted students remain passive. To prevent this, the teacher could form teams with only equally gifted students. Another, maybe more useful approach is to equally distribute all abilities within the groups. If the presentation of the teamwork has to be done by a randomly chosen team member, the more gifted students will help those less gifted – in particular if the result of the teamwork counts as a performance for all team members. Such peer teaching is advantageous to all students: the less gifted students receive support, while the more gifted students learn further through their teaching. Teamwork also helps improving communication skills (and not only regarding mathematics).

☯

Generally, the use of technology often is a trigger for group work as well as for *oral mathematics*, i.e. the communication about mathematics.

A good mathematics teaching and learning tool should support students communicating with each other and with the teacher. Exchanging documents and screen content would be desirable for facilitating remote forms of teamwork with team members not sitting next to each other.

☯

Compensate

Say, your students have to solve the equation

$$x + 6 = 18 - 2x$$

Solving such an equation for *x* is done by transforming it into the form "x = term with no x". This is achieved through choosing and applying an appropriate sequence of equivalence transformations. Typically one will "*bring terms with x to one side of the equation*" and "*bring all other terms to the other side*". Therefore a good first choice is to add $2x$ to both sides of the equation.

$$x + 6 = 18 - 2x \quad \mid +2x$$

After *choosing* this equivalence transformation, we have to *apply* it to both sides of the equation i.e. we have to simplify the two expressions $x+6+2x$ and $18-2x+2x$.

$$\begin{aligned} x + 6 &= 18 - 2x && \mid +2x \\ 3x + 6 &= 18 \end{aligned}$$

Now it would be appropriate to subtract 6 from both sides.

$$\begin{aligned} x + 6 &= 18 - 2x && \mid +2x \\ 3x + 6 &= 18 && \mid -6 \\ 3x &= 12 \end{aligned}$$

We are interested in the *practice* of teaching and learning mathematics. There is no need to care about the students who succeed – because what can we do better for them? We should care about the students who don't succeed. We should strive to find out why they make certain errors and how we can help them to avoid these errors.

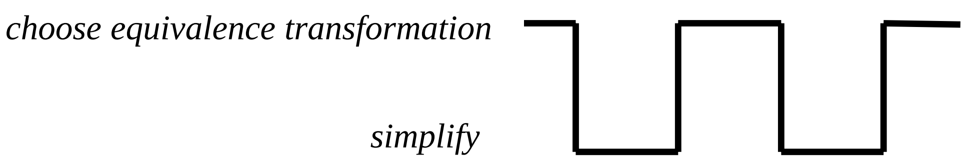
Back to the equation $3x = 12$. At this point some students find it difficult to choose a good next step. The following argument is quite typical for many students: *"There is a 3 in front of the variable x. To get rid of the 3, I must subtract 3."*

A student, who uses this argument in a paper and pencil environment, most likely will proceed as follows:

$$\begin{array}{rcll} x+6 &=& 18-2x & \mid +2x \\ 3x+6 &=& 18 & \mid -6 \\ 3x &=& 12 & \mid -3 \\ x &=& 9 & \end{array}$$

The student will transform the equation $3x = 12$ into $x = 9$ and believe that the equation is solved. It will take the student quite a while to determine that a mistake has been made and even longer to find out what was wrong.

What goes wrong and how can technology help to make it better? An analysis of the steps taken above reveals two alternating tasks: (1) the choice of an equivalence transformation and (2) the simplification of algebraic expressions. Here, the choice of an equivalence transformation is a higher-level task insofar as it is the essence of the strategy for finding the solution of an equation. It is the new skill that the student has to learn when learning to solve equations. The simplification of expressions is a lower-level task, for which the teacher has to assume that the student is sufficiently well trained.



This picture demonstrates that a student, while trying to learn the new skill, repeatedly has to interrupt the learning process in order to perform a simplification. This is as if one would repeatedly be interrupted during a difficult chess game. In fact, it is even worse, because the interruption can influence the „game“: A mistake made during the interruption, i.e. while performing the lower-level task, severely disturbs the higher-level task and may prevent the student from learning. This is exactly what led to the wrong solution $x = 9$ in the above example: After deciding to subtract 3, ideally the student should fully concentrate on subtracting 3 from both sides of the equation while “forgetting” the motivation for choosing this equivalence transformation. But, in reality, the student starts the next line with “*x* =” simply “*because the transformation -3 was chosen in order to generate ‘x =’ on the left hand side*”. But then, at the higher level, the student has the (wrong) impression that -3 simplified the equation as desired.

This continuous change of levels inevitably occurs in almost all topics in school mathematics. It appears to be one of the central problems in mathematics education that students have to learn a new ability/skill while still practicing an “old” one.

In the sequel we demonstrate how one can use TI-Nspire CAS to help students in this situation.[14]

Enter the equation $x+6=18-2x$.

```
x+6=18-2·x                                x+6=18-2·x
```

Add 2*x* to both sides of the equation by typing: +2x

```
Ans+2x
```

Plus is a binary infix operator – and because it was entered without a first argument, a reference to the last answer, Ans, was introduced.

Conclude the input with the Enter key.

```
(x+6=18-2·x)+2·x                          3·x+6=18
```

The resulting entry-answer pair shows the equation and the unsimplified equivalence transformation on the left and the resulting equation on the right. The next step is to subtract 6 from both sides. To do so, type ‘−6’:

Start with typing a minus: -

```
Ans –
(–) Negate                                   37/99
```

Because there was no first argument, the input is ambiguous. With this selection menu TI-Nspire CAS requests to choose the meaning of the minus, as there are two types of minus: an infix binary minus, called subtraction, and a unary minus, called negation. The first choice in the selection menu is the subtraction minus. The text indicates that a reference to the last answer, Ans, will be inserted before the minus.

Use the Enter key to confirm the highlighted subtract minus or click on it.

```
Ans–
```

Enter: 6

```
(3·x+6=18)-6                              3·x=12
```

[14] The original source of this approach is a paper by Aspetsberger/Funk, see *References*.

So far everything is like it was with paper and pencil. Now we mimic a student who chooses to subtract 3. See what happens in TI-Nspire CAS if you subtract 3 from both sides.

Enter: `-3`

$(3\cdot x=12)-3 \qquad\qquad 3\cdot x-3=9$

Clearly, the tool applies the equivalence transformation correctly. Therefore the student *immediately* sees that subtracting 3 did *not* simplify the equation as expected. Instead, it complicated it. TI-Nspire CAS gave important immediate feedback on the quality of the student's choice. It is like putting the finger on a hot stove and feeling the pain *immediately*. This is a good prerequisite for successful learning. Students can concentrate on finding suitable equivalence transformations without being hindered by a possibly (still) poor simplification skill. The above is a practical example of using technology as a compensation tool.

Undo the last step, and then try dividing by 3:

Undo the last step with the **Undo** button.

$(3\cdot x=12)-3$

You are back in expression input mode. Change the minus operator to a division operator:

Replace '−' by '/'.

$(3\cdot x=12)/3$

Conclude the input.

$\dfrac{3\cdot x=12}{3} \qquad\qquad x=4$

This educational approach is called the scaffolding method. It offers students essential support for building more advanced mathematical knowledge even though they might not have mastered some prerequisites. Some of these skills might be needed only for technical reasons, being unnecessary for understanding the more advanced concept. Thus, technology plays the same role as a scaffolding for building a house while some of the lower stories are still incomplete. This metaphor is the reason for the name *scaffolding method*. The idea is based on what Bruno Buchberger in the mid 80s suggested as the "Black-Box-White-Box Principle".[15]

[15] See the paper by Buchberger in the *References*.

We look again at the picture with the two tasks that one has to alternatively concentrate on when solving equations with paper and pencil. When the computer takes over the simplification task, as was done in the above exercise with TI-Nspire CAS, …

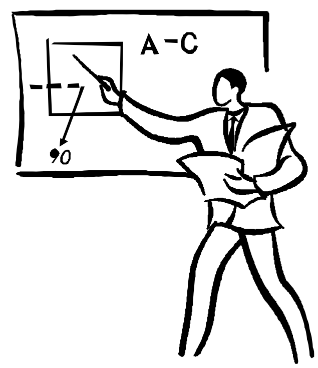
choose equivalence transformation

simplify

… the students can fully concentrate on the higher level task.

☯

Here is another example. Say, we have used ample time to teach and practice how to solve systems of linear equations. At some point in time we do have to move on to the next topic, simply because we have to fulfill a teaching schedule. At this point some of our students will have mastered the solving of systems of linear equations while others will have not.



Say, the next topic is analytic geometry. Many analytic geometry problems require the solving of systems of linear equations. So what about those students who still struggle with systems of linear equations? They will find it difficult if not impossible to solve most of the analytic geometry problems!

For a moment we go back to the optical instrument metaphor from the *Introduction*: for safety reasons a good eyesight is a prerequisite for being allowed to drive a car. What about people with poor eyesight? Should they be banned from the road traffic? There is no need to, because they can (and must) use *eye-glasses* that make up for their weakness.

Accordingly, we should allow students with a poor solving-systems-of-linear-equations skill to use a compensation tool when “driving in analytic geometry land”. In fact, this is not only an act of humanity, but this is our pedagogical duty! Banning technology from the classroom and forcing “mathematically challenged” students to do analytic geometry without a much needed compensation tool is like banning eye-glasses from the road traffic!

It goes without saying that we should strive to remove any weakness that we find with a student. But we need to distinguish between “therapy” and “routine work”.

Analytic geometry should not be (mis)used as a therapeutic opportunity to repair a solving-systems-of-linear-equations weakness!

☯

The following analogy from dancing clarifies the scaffolding concept even further.

Viennese Waltz is pretty simple – at least from a mathematical point of view. One has six beats of the music to move forward six steps while doing a 360 degree turn. The theory of Viennese Waltz sounds simple; the practice is more challenging when trying to follow the rhythm of a piece of music and even more so when trying to do this with a partner in continuous body contact.

Teaching Viennese Waltz usually starts with asking the students to stand so that they look into the direction that they want to (are supposed to) dance. Starting with the right foot forward they should do a 180 degree turn with three steps on the three beats 1 – 2 – 3 (one step per beat). After completion of these three steps they should be looking at the position they were coming from.

We will ask our students to practice this small three step routine for a while. After some time, usually, there will be two groups of students: Those who can turn 180 degree on three steps – and those who cannot. “Without loss of generality” (and for simplicity) let’s assume that the second group achieves only a 90 degree turn. For later reference we will label these two groups of students *180-degree-students* and *90-degree-students*.

We assume now that the next exercise would require everybody to do a full 360 degree turn on the six beats 1 – 2 – 3 – 4 – 5 – 6. If we ask the students to try this, then the 180-degree-students may be able to do it, but the 90-degree-students would be lost completely. And it is clear, why: in order to succeed in the end, a 90-degree-student would have to make up for the (known) poor performance in the first three steps by doing a 270 degree turn on 4 – 5 – 6. But this is a real challenge even for a good dancer!

How can we do better in teaching the Viennese Waltz turn? We ask the students to stand with their backs into the direction that they want to (are supposed to) dance. In other words, we ask them to *pretend* that they just did a perfect 180 degree turn on 1 – 2 – 3, independent of whether they can or cannot. Then we ask them to do, starting with the left foot backward, a 180 degree turn with three steps on the three beats 4 – 5 – 6 and let them practice this for a while.

The next phase would be to combine 1 – 2 – 3 and 4 – 5 – 6. Then we add music. Then we add a dancing partner.

This is exactly the idea of the scaffolding approach: One pretends that one can do all lower level tasks by delegating them to the tool. This allows to fully concentrate on the new, higher level task.

☯

Here is a quote from a teacher who made an observation in his classroom after he learned about the scaffolding method:

I had a simple example today of a boy
who was dropping behind in algebra
because he was struggling to cope with the mental arithmetic,
which he saw as a vital skill for the exercise.
Gently persuading him to use a calculator made quite a difference
and he was able to demonstrate
that he had good competence in the algebraic skills.
The different levels of use of mathematics are really applicable
at any time in the classroom.
(Peter Ashbourne)

Solve

Traditional mathematics teaching is very much centred on solving problems – ranging from simple calculations such as 5+12 = ? or $3x+4x$ = ? to complex word problems involving optimization.

Technology for supporting one of the other five archetypes (*Represent*, *Document*, *Communicate*, *Compensate*, *Explore*) is mostly seen as a supplement or as an enrichment of traditional teaching. Technology for solving problems, however, by many teachers is seen as a competition for what they do in the classroom or even as a threat to the students.

Computer algebra systems provide a rich collection of black boxes for solving problems in algebra, trigonometry, calculus, matrices, and other areas. Popular commercial CAS automate up to 80 percent of what we teach until the end of high school (with exit exams such as “Abitur”, “Matura”, “Baccalaureate”, or “A Level”). This is the reason why the appearance of CAS has shaken mainstream mathematics teaching all over the world.

Computer algebra systems polarize educators into supporters and opponents. Many supporters would like to use CAS whenever possible, because this would allow for solving more (realistic) problems in the classroom. Many opponents would like to ban CAS, because they believe that scientific calculators destroyed their students’ mental arithmetic, which is seen as a vital mental skill, and CAS could have an even more devastating effect by destroying mental algebra, mental trigonometry, mental calculus, etc.

Both arguments appear plausible – so what shall we do? Which mental faculties do we need – and how much of each?

☯

How far can you see? How much can you hear? How loud can you shout? How far can you reach? How far can you walk? How much math can you do? We have many horizons, each being defined by a faculty that we possess. The faculty of hearing defines the audio horizon, the faculty of seeing defines the visual horizon, etc. The faculty of performing arithmetic, algebra, trigonometry, calculus, etc. defines the horizon of the mathematical problems that we can solve.

Throughout history people tried to extend their horizons by making intelligent use of nature and/or by building amplification tools. A megaphone (most simply formed with the two hands around the mouth) increases the reach of the voice. An ear trumpet (most simply formed with the hands extending the outer ears) allows for better hearing. With a horse we can move faster and greater distances. More recent moving tools are bicycles and cars. With a telescope we can see further than our natural eyesight would allow. And so on. Today we even have tools that allow us to do something that we cannot do naturally – such as fly.

Pebbles helped early history people with their arithmetic. Later, a “user interface” was added by arranging pebbles into an abacus. Today’s computers are much advanced abaci as was already discussed in an earlier chapter.

The history of mankind is a history of producing tools and technology.

☯

With any kind of technology a key question is when to use it and when not to use it. When we have a car available should we use the car whenever we want to go from A to B?

If A and B are a hundred kilometres apart, the answer is “yes”. If A and B are only five meters apart, the answer is “no”. What conditions make the “no” turn into a “yes”? Is it just a distance? Or is it (also) a purpose – because every desire to move from A to B has a purpose. Are any other issues relevant for this decision?

Physical fitness certainly will be an issue here. A physical challenge such as a handicap of walking will influence the decision. This is what we already discussed in the chapter *Compensate*.

If A and B are three kilometres apart – should we walk or drive? If the purpose for moving from A to B is to do some shopping, then going by car appears reasonable, in particular because we may not be able to carry all the groceries that far back home. If the purpose for moving from A to B is to improve physical fitness, then we should jog – not drive.

This thinking can be applied also to using mathematical tools. As an example, we look at the function (or button) solve, which is a “black box” for solving many types of equations, systems of equations, inequalities, and systems of inequalities. When should we use solve?

When we ask our students to solve an equation, there are two possible motivations for that. Either we want the solution – for example, because the solution is needed within a bigger context such as an analytic geometry problem – or we want the students to take the steps to the solution so that they develop or improve their (mental) algebra skills. This is exactly as it is with physical movement: When we move, then either we are interested in reaching the destination or we are interested in the moving. The key question in the classroom, therefore, is:

*Are we interested in the **solving** or in the **solution**?*

When we want the *solution*, then we should use technology so that we obtain the solution quickly and can rely on its correctness. This serves the problem solving skill in the best possible manner, because we (as well as the students) can fully concentrate on the strategy of solving the bigger problem rather than concentrate on performing the necessary calculations.

When we want the *solving (process)*, then we should not use technology (except, when necessary, for lower level tasks as was explained in the chapter on *Compensation*).

Here is a reverse thought: If we want our students to practice the solving of systems of linear equations, we should give them systems of equations to solve. We should not *abuse* higher level topics such as analytic geometry for that!

The following equality provides a useful model:

(school) mathematics = mental training + problem solving training

Educators who desire to ban technology are advocates of mental training. Educators who desire to use technology as much as possible are advocates of problem solving training.

Nothing is either good or bad – only thinking makes it so.
(William Shakespeare)

School mathematics has both aspects and we should have or create room for both in the classroom.

☯

Mental training has never been as important as it is today.

Around 1750 the steam engine was invented. With this tool, people could create power as and where needed from any flammable substances such as wood or coal. Both the ease of generation and the amount of power that could be generated made this a true quantum leap. The steam engine led to unimaginable possibilities: the *industrial age* had begun. Even today we are still amazed at what the steam engine and its successors (bulldozer, ocean liner, aeroplane, spaceship, etc.) can do.

Before the industrial age, one had to use one's body to earn one's daily bread. Today that is no longer the case. However, most people realise that the body needs exercise so as not to fall into ruin. This is why so many people in the industrialized parts of the world now take part in recreational sports such as jogging, aerobics, body-building and skiing in order to keep fit.

Around 1950 the computer was invented. With this tool, people could create intellectual force (so to speak) as and when needed, at first principally in terms of memory and numerical calculation power. This invention caused another new quantum leap. With the computer, in particular with the possibilities of modern telecommunications, totally new possibilities arose: the *information age* had begun.

Up to the information age most people *had* to use their intellect. In the future fewer people will be required to[16], and will thus realise that the intellect needs to be exercised so as not to fall into ruin. “Thought sports” may well become as popular in the twenty-first century as jogging was at the end of the twentieth century. That this development is already happening is obvious from the sharply increasing sales figures of specialist books and (computer) games as well as from the popularity of TV quiz shows.

☯

Mathematics is the principal means of educating the human mind.
(Carl Friedrich Gauss)

Sometimes one sees in the school simply the instrument for transferring
a certain maximum quantity of knowledge to the growing generation.
But that is not right. Knowledge is dead; the school, however, serves the living.
It should develop in the young individuals those qualities and capabilities
that are of value for the welfare of the commonwealth.
But that does not mean that individuality should be destroyed
and the individual become a mere tool of the community, like a bee or an ant.
…
If you have followed attentively my meditations up to this point,
you will probably wonder about one thing.
I have spoken fully about in what spirit youth should be instructed.
But I have said nothing yet about the choice of subjects for instruction,
nor about the method of teaching.
Should language predominate or technical education in science?
To this I answer:
In my opinion all this is of secondary importance.
If a young man has trained his muscles and physical endurance
by gymnastics and walking, he will later be fitted for every physical work.
This is analogous to the training of the mind
and the exercising of the mental and manual skill.
(Albert Einstein – from a speech to educators in 1936)

☯

[16] With cell phones we don’t need to memorize phone numbers. With navigation systems we don’t need to use our sense of direction. And so on …

Computer algebra systems force us to ask questions that we should have asked earlier. We did not – and now we must. The above exposition gives an answer to the question *"what is the purpose of a classroom task?"*

☯

Another question is *"what are indispensable manual skills?"* What manual calculation skills are still needed when students use numeric, graphic, or algebraic technology? What should students be able to do manually, i.e. just using paper and pencil? In many countries this question now is discussed under the title "Standards of Mathematics Education".

There is no general answer to this question. Using the car metaphor, this is the question as to what distance the students should be able to move without using a car. Ultimately, the answer is a matter of definition.[17]

☯

Assessment is an important pedagogical instrument. Therefore it is logical to ask *"how to integrate technology into assessment"*. Naturally, this question is tightly connected with the question about the standards, because whatever we declare an indispensable manual skill we need to test as a manual skill, i.e. in a technology-free environment.

A practical answer is easily derived from the "mathematics = mental training + problem solving training" model. One simply splits the exam in two parts: When assessing mental fitness, no tools are allowed. This includes even a simple four-function calculator. When assessing problem-solving capabilities, all tools are allowed – or, better, solicited. This includes graphing and algebraic tools such as computer algebra systems. If the split is not manageable within a single exam, one should assess the two "disciplines" at different times.

Here is a parallel with ice skating: Mental training compares with the compulsory exercise, in which the athlete demonstrates a mastery of the required basic techniques. Problem solving compares with the voluntary exercise (= freestyle), in which the athlete demonstrates the ability to combine the basic techniques into a choreographed presentation. The total score depends on the scores of both the compulsory and the voluntary exercise.

From the world of teaching, foreign language teaching may serve as an example, because dictionaries are well integrated "tools" for teaching and learning a foreign language.

A good skill in a foreign language comprises two sub-skills: one has to know enough words (their syntax and their semantics) and one has to be able to com-

[17] A provoking attempt is in the paper by Herget/Heugl/Kutzler/Lehmann, see *References*.

bine the words into meaningful text. The vocabulary compares to the indispensable manual calculation skills and the writing of text compares to the problem solving skill.

In foreign language teaching these two skills are assessed in two different tests. Naturally, dictionaries are not allowed in word tests, while (normally) they are allowed when students have to write a composition.

☯

We should make sure that students have similar (if not to say equal) chances. There are many different kinds of mathematics technology on the market – with prices ranging from “affordable” to “exclusive”. Will students who can afford to buy expensive tools have an advantage? Using the car metaphor this question translates into *“does the Porsche owner have an advantage over someone with an inexpensive economy car?”*

For an answer we again look at the example of foreign language teaching, where there are many different kinds of dictionaries on the market.

A good foreign language test is one for which the quality of the dictionary does not make (or hardly makes) a difference in the test’s outcome. When writing a composition, the emphasis should be on everything that the dictionary cannot help with.

Essentially, a dictionary, no matter if manual or electronic, plays – or should play – only a minor role in foreign language assessment. And this is exactly the lesson that we should learn for the integration of technology into mathematics teaching and learning in the long run.

We should develop a teaching, learning, and assessment culture in which the questions that we ask, the problems that we pose, and the way that we evaluate the answers and results do not depend (or hardly depend) on the technology that is used in the exam. Technology shows us that the performing of calculations is the least important part of mathematics.

Mathematics is the art of avoiding computations.
(Bruno Buchberger)

Computers and calculators should be for mathematics teaching and learning what dictionaries are for foreign language teaching and learning. Not more and not less.

☯

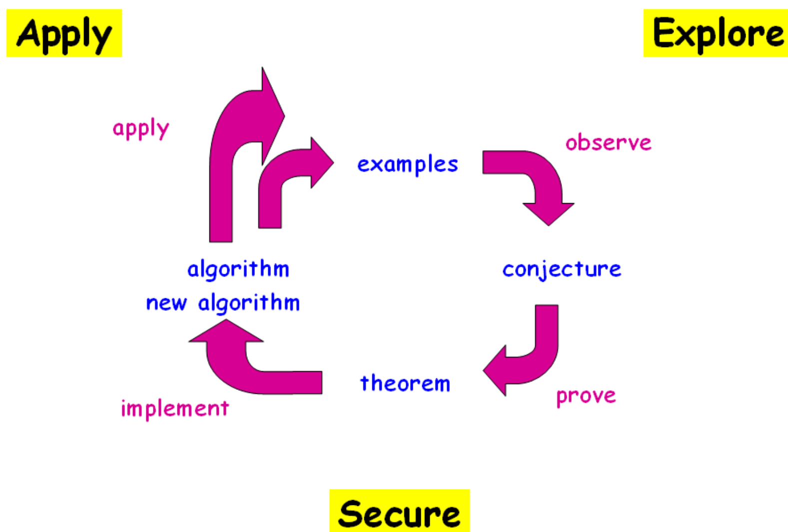
If it is not necessary to use technology,
then it is necessary to not use technology.
(Helmut Heugl)

Explore

How do we learn walking, speaking, riding a bike, dancing, …? We learn by doing. We learn by trial and error. We learn by exploration. We try, we observe, we fail, we analyze, we try again, …

How did mankind discover all the mathematics that we know today and how do we find even more mathematics? By the very same method.

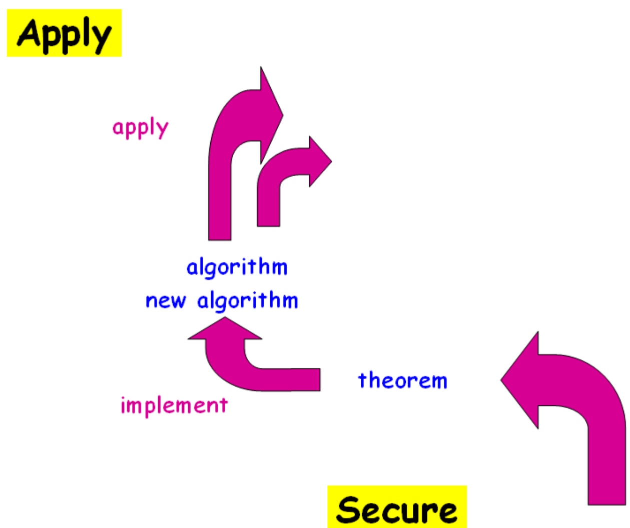
More formally, we can describe the method of mathematical “growth” as follows. Applying known algorithms produces *examples*. From the examples we *observe* properties that are inductively expressed as a *conjecture*. Proving the conjecture yields a *theorem*, i.e. guaranteed knowledge. The theorem‘s algorithmically usable parts are *implemented* in *new algorithms*. Then the old and the new algorithms are *applied* to new data, yielding *new examples* that lead to new observations, new conjectures, and so on.



This picture of a spiral that demonstrates the path of discovery of (mathematical) knowledge was proposed by Bruno Buchberger. In the spiral we find three phases: the phase of *exploring*, the phase of *securing*, and the phase of *applying*. These three phases can also be denoted as **in**duction, **de**duction, and **pro**duction.

☯

In its beginnings mathematics was a purely experimental science, i.e. it consisted only of the phases of exploration and application. Then the Greek applied to it the deductive methods of their philosophy (i.e. they added the phase of securing), thus establishing mathematics as the deductive science as we know it today. Fairly recently (in terms of history – notably in the first half of the 20th century) a group around the French mathematician Jean-Alexandre-Eugène Dieudonné (the group became known under the name *Bourbaki*) restructured the mathematical knowledge using the system of “definition-theorem-proof-corollary-example ...”. This Bourbaki system, also called *Bourbakism*, being developed for the purpose of inner-mathematical documentation and communication, comprises only the phases of securing and applying and has become characteristic to modern mathematics. All mathematics research is published in this style. But then Bourbakism gradually lodged itself in teaching and learning. It has become customary to teach mathematics by presenting mathematical knowledge, and then asking the students to learn it (= secure it) and apply what was learned to solve homework and exam problems.



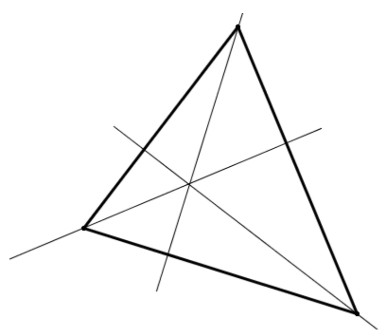
Once we have finished the two phases for a certain topic, then we start over with presenting the next topic; and after that the next topic, and so on. But there is no spiral any more. There is only a sequence of repeated presenting-learning-applying phases.

This is a highly unnatural way to (try to) learn. No mathematician could do mathematical research the way we demand our students to do it. Mathematicians do go through the full spiral.

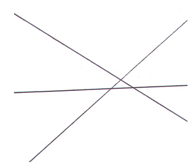
Probably it is the available Bourbaki style mathematical documentation (that nowadays also includes mathematics textbooks) that gives the wrong impression that mathematics is not an “experimental science” although it definitely is – to some extent. A good example is Andrew Wiles’ proof of Fermat’s Last Theorem. Andrew Wiles worked for about seven years on this proof, and obviously he spent most of the seven years in the phase of exploration. The Bourbaki style summary of his work is a 109 page article in the journal *Annals of Mathematics* that may have taken him several weeks to write – still only a very small portion of the seven years …

A student has to „locally“ build his individual little „house of mathematics“ while a scientist does pretty much the same „globally“ by trying to find mathematical knowledge that is new for mankind (whereas for the student it is new “only” for him or her). For both the scientist and the student a substantial part of knowledge acquisition happens during the phase of exploration. From this point of view it becomes understandable why so many students are at loggerheads with mathematics, and one will demand that exploration obtains its due position within the teaching of mathematics. Phases of exploration should complete the traditional teaching methods – not substitute them! This is not a plea for returning to pre-deductive Egyptian mathematics but a plea for mathematics teaching and learning going through all three phases of the knowledge acquisition spiral.

However, it is understandable that, within the framework of today‘s curricula, there was hardly any exploration in the mathematics classrooms. Exploration, performed with paper and pencil, is both time consuming and error prone. Within the time available at school, students can generate only a very small number of hand produced examples for the purpose of observing and discovering, and a hefty portion of these examples probably would be faulty due to calculation and other errors. There is nothing that you can observe from only a few, partly wrong examples!



Look at a typical example from geometry. Say, we want to teach our students that in every triangle the three altitudes intersect in one point. We might ask them to draw five triangles, and then construct the three altitudes in each. But what happens? Most of our students – being lousy drafts(wo)men – will find that in three or four of their five triangles the three altitudes do NOT intersect in one point. And this should convince them that this is a true statement?!



From now on technology enables students to experiment within almost all topics treated in mathematics teaching. Students can use tools such as computer algebra systems, dynamic geometry systems, and spreadsheets for doing large numbers of examples in a short time and the electronic assistant guarantees the properness of the results. Talking about an assistant: historic records indicate that great mathematicians such as Carl Friedrich Gauss employed herds of human “calculators” without which they would not have made most of their famous findings.

The plea for allowing students to find what they are supposed to learn is not new.

Help me to do it by myself.
(Maria Montessori)

While Montessori pedagogy is successfully used for lower level education it was not yet possible to use it for high school mathematics. Latest computer technology, in particular CAS, allows for that.

We should not teach students something that they could discover themselves.
(Hans Freudenthal)

It took many hundreds of years and very many great minds to discover the mathematics that our students are supposed to learn today. It is presumptuous to believe that they can make these discoveries all by themselves – even with technology. For sure they won’t stay at school for however long it would take them for that.

With technology we can implement a new teaching and learning culture that could be called *guided explorations*, in which the teacher observes the students in their experiments and feeds them with useful hints along their “explorative journey” in order to help them reach the expected goal, i.e. make the intended discoveries.

☯

Give a person a fish and you feed them for a day.
Teach a person to fish and you feed them for a lifetime.
(Confuzius)

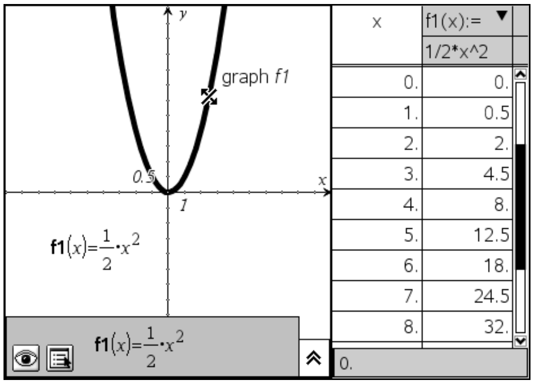
By translating this quote of Confuzius into the language of teaching and learning mathematics, we get a very good description of what we have and what we should try to achieve:

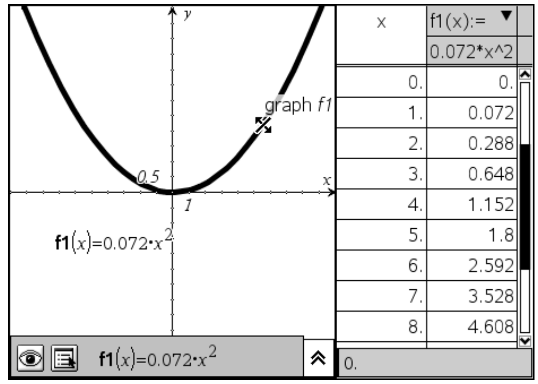
Give a student some mathematics and you feed them for the next exam.
Teach a student to fish for mathematics and you feed them for a lifetime.
(Confuzius, adapted)

☯

Dynamic geometry systems such as Cabri Geometry (also included in TI-Nspire), Geometers Sketchpad, Cinderella, GeoGebra, and Autograph are typical tools for explorative learning in Euclidean and analytic geometry.

In the chapter *Represent* we showed screens of TI-Nspire, in which one can grab a graph, and then move the graph and see how the corresponding function expression changes.





This is a good exercise for experimenting, discovering and understanding how the factor of x^2 influences the shape of the graph.

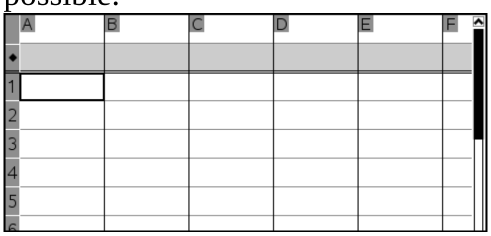
Mathematics is the science of patterns.

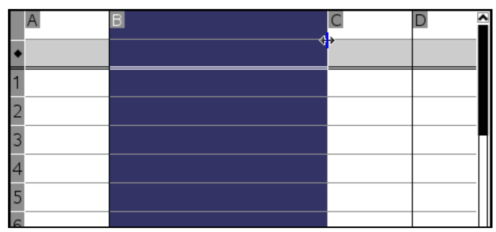
Technology helps to generate enough examples for the students to be able to see the patterns.

☯

Following is a session with TI-Nspire CAS, in which we create a spreadsheet that experimentally reveals patterns of differentiation for discovery.

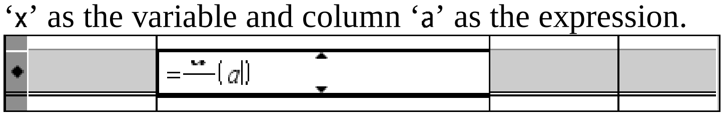
- Open a Lists&Spreadsheet page, and then make the second column as wide as possible.

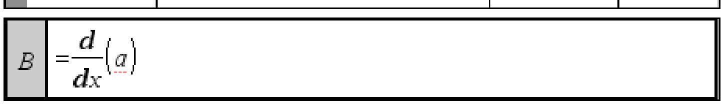




Define the second column to be the derivative of the first column with respect to x:

- After entering the equal sign, paste the derivative template, and then choose ‘x’ as the variable and column ‘a’ as the expression.





Conclude the input.

	A	B	C	D
◆		=**d**(a□,x)		

You are ready now to make a nice pattern recognition exercise with your students. The goal should be not too ambitious, but not too easy either. To start with, let them find the rule for the derivative of the *n*-th power of *x* by showing them some examples:

Enter into the first column the expressions x^2, x^5, and x^9.

	A	B	C	D
◆		=**d**(a□,x)		
1	x^2	2*x		
2	x^5	5*x^4		
3	x^9	9*x^8		

Ask your students if they see a pattern and let them describe their findings. You may want to add another example:

Enter x^{25} into the first column.

4	x^25	25*x^24		

Probably your students will have the correct answer by now. But you should not make it that easy for them. Challenge them by entering a negative power of *x*:

Enter x^{-4} into the first column.

5	x^(-4)	-4/x^5		

At first glance this seems to not fit the pattern. Let your students recollect what they now about powers. If needed, help them remember $\frac{1}{x^5} = x^{-5}$.

Next, challenge them with a fractional power:

Enter $x^{-\frac{4}{3}}$ into the first column.

6	x^(-4/3)	-4/(3*x^(7/3))		

The next challenge could be a fractional power for which TI-Nspire CAS uses a special notation:

Enter $x^{\frac{1}{2}}$ into the first column.

7	x^(1/2)	1/(2*√(x))		

Again, this seems to not fit the pattern. But probably, encouraged by the previous examples, your students will look for ways to rewrite the expression appropriately. If needed, help them remember $\sqrt{x} = x^{\frac{1}{2}}$.

Finally challenge them with the “hidden” first power of *x*:

Enter *x* into the first column.

8	x	1		

As simple as this looks, this may be the hardest challenge, aiming at remembering both $x = x^1$ and $1 = x^0$. At the end of this exercise your students have discovered the rule $\frac{d}{dx} x^n = n \cdot x^{n-1}$ and they have reinforced some important facts about powers.

☯

The above template for derivatives can be used to discover many more differentiation rules, including the chain rule.

Enter $(2x+5)^3$ into the first column.

10	(2*x+5)^3	6*(2*x+5)^2			

Enter $(x^2+1)^3$ into the first column.

11	(x^2+1)^3	6*x*(x^2+1)^2			

Let the students discover what the difference is between the expected factor 3 and the actual factor.

☯

When students discover a rule by observing a pattern, it is much easier for them to remember the rule when they need it, because the path that led to the discovery left a trace in the brain.

☯

More Thoughts about Teaching and Technology

Being a teacher is something very special. It goes far beyond possessing a good faculty of something. Someone with a good faculty of reading not necessarily can teach a child how to read. Also, not every good sports(wo)man later turns into a good coach. Good teaching is a very fine art!

And it is more than that.

Sometimes the wealth of a country is measured by the amount of mineral resources that it possesses such as oil, copper, silver, gold, or diamonds. But all of these resources are finite. There is a much more precious resource that all countries have: humans.

Youth is the wealth of a nation.
(Sheikh Zayed, former ruler of United Arab Emirates)

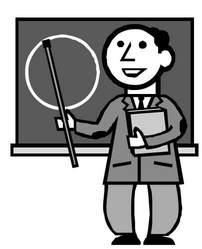
Teaching is the art of developing human resources. Therefore, teaching greatly contributes to the wealth of a nation.

Teachers help the country to develop human capital.
(Star – The People's Paper, Malaysian newspaper, Monday 17 May 2004)

In the past the development of mineral resources was achieved by human labour using simple tools such as shovel and staple. Efficiency was multiplied by using latest technology such as caterpillars and drilling derricks.



In the past the development of human resources (= teaching) was achieved by teachers using chalk and blackboard. Also here efficiency can be multiplied by using latest technology such as computer algebra systems, dynamic geometry systems, and spreadsheets.


Traditionally we use our intellects when we teach and we address our students' intellects. However, feeling is much more important than thinking. Therefore, feelings are a very effective support for teaching and learning.

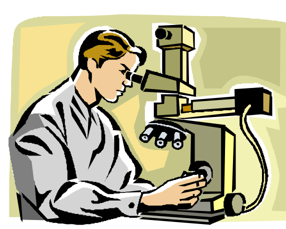
Experience shows, that students can get very excited when using computers and calculators for making discoveries. Therefore, technology supports the teaching through emotions.

Here is one more picture that helps to understand the benefit of using technology in the classroom.

A mathematics teacher is like a tour guide who has to guide a group of hikers, comprising top athletes and physically challenged persons, through rough terrain such that everybody arrives in good mood and at the same time at the final destination.

This is exactly the situation that we face in a typical mathematics classroom with both mathematically gifted and "mathematically challenged" students.

With technology we can master this situation in the best possible manner. For the mathematically challenged students technology is a compensation tool (a wheel chair, a crutch) with which they can move faster. For the mathematically gifted students technology is an exploration tool that "entertains" them or keeps them busy with fascinating discoveries while they have to wait for the others to catch up.



Casanova or Don Juan?

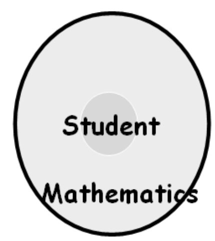
Giacomo Girolamo *Casanova* was an Italian adventurer and writer who lived 1752-1798. He had a degree in law, but also had studied mathematics. *Don Juan* is a legend, used as hero in opera, play, and fiction. The first written version was published in Spain around 1630.

Both are famous for being womanizers – though there is a significant difference: Casanova wanted pleasure for the women, Don Juan wanted pleasure for himself.

We can use this difference for a classification of teachers, notably

- **Casanova-type teachers** and
- **Don Juan-type teachers**.

There is no teaching without a student. Therefore, the student is (or should be) in the centre of all teaching.



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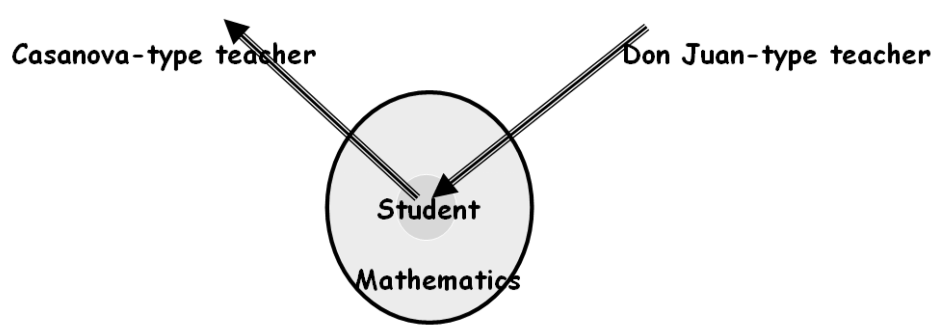
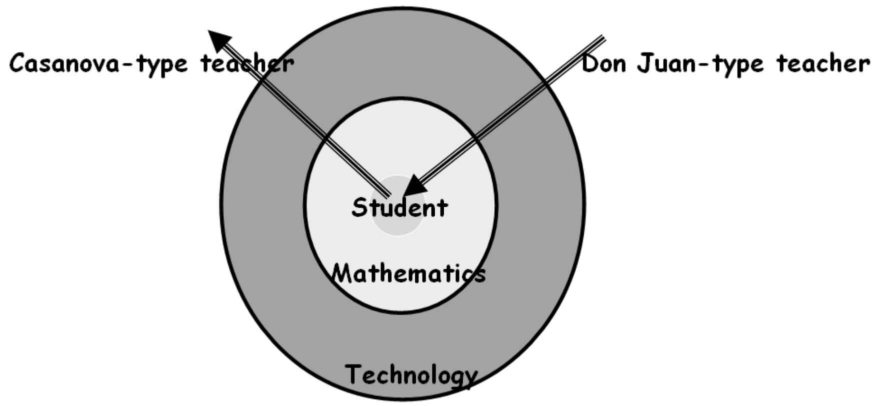
A Casanova-type teacher meets the student where he or she is and guides him or her through the topic of teaching as far as this student can go. The student comes first in this endeavour and mathematics comes second. For a Casanova-type teacher every (group of) student(s) is a new challenge and the teaching is always different.

For a Don Juan-type teacher mathematics comes first and the student comes second (if not to say 'last'). Typically their teaching is always more or less the same, notably a "sink or swim" style.



In essence, a Casanova-type teacher *teaches students* and a Don Juan-type teacher *teaches mathematics*.

When using technology, the difference between these two types of teachers may become even more dramatic:



For a Casanova-type teacher the student still is in the centre, mathematics is secondary and serves the development of the student, and technology is tertiary and serves the dissemination of mathematics.

If a Don Juan-type teacher is a fan of technology, then technology may become his or her primary interest of teaching, so that they end up *teaching technology*.

☯

As said before, students are the original goal of all teaching.

Therefore, we should teach students.

Therefore, we should be Casanovas.

After all that you have read in this book you will understand my plea that …

… *we should be **CAS**anovas.*

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